A Variational Formulation of Teichmüller's Modulsatz and an Application to Pointwise Conformality

Melkana Brakalova-Trevithick[1]

[1]Fordham University, New York, USA

Abstract

Extremal problems in conformal geometry provide a rich source of variational questions. In this talk, we reformulate in variational language a less well-known geometric distortion problem investigated by Teichmüller in his habilitation work [1] (see also the English translation by the author and Weiss [2]), and apply it to the study of pointwise regularity of quasiconformal maps.

We first discuss this new variational form of Teichmüller's Modulsatz and its proof, in which one studies a one-parameter family of sharp threshold problems.

Let $\mathcal{K}$ be the class of compact separating sets $K \subset \widehat{\mathbb{C}}$ such that $\widehat{\mathbb{C}} \setminus K = \Omega_0(K) \cup \Omega_\infty(K)$, where $\Omega_0(K)$ and $\Omega_\infty(K)$ are simply connected domains containing 0 and ∞, respectively, with conformal radii $r_0(K)$ and $r_\infty(K)$. For $0 < \varepsilon < \log 4$, set $B_\varepsilon(K) = \{w : \; r_0(K)e^{-\varepsilon} \le |w| \le r_\infty(K)e^{\varepsilon}\}$, and define $\delta(\varepsilon) = \sup\{d > 0 : \; \log(r_\infty(K)/r_0(K)) < d$ implies $K \subset \operatorname{int} B_\varepsilon(K)$ for every $K \in \mathcal{K}\}$. Teichmüller's work provides an ingenious construction of an extremal compactum K_ε for which containment just fails and $\delta(\varepsilon) = \log(r_\infty(K_\varepsilon)/r_0(K_\varepsilon))$. To the best of our knowledge, the extremal compacta K_ε themselves have not been studied in the literature, although related proofs of Teichmüller's asymptotic estimate $\delta(\varepsilon) \sim \varepsilon^2/\log(1/\varepsilon)$ as $\varepsilon \to 0^+$ appear in research monographs such as those of Pommerenke [3] and Garnett–Marshall [4].

We apply these methods to improve a $C^{1+\alpha}$-conformality theorem of Shishikura [5] on quasiconformal mappings at a point. The extremal construction of K_ε suggests that the resulting improved pointwise conformality estimates may be sharp.

This presentation provides a bridge between a variational formulation of a classical conformal-geometric extremal problem and modern pointwise regularity questions for quasiconformal mappings.

References

[1] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildungen*, Deutsche Mathematik **3** (1938), 621–678.

[2] O. Teichmüller, *Investigations on conformal and quasiconformal mappings*, translated by M. Brakalova-Trevithick and M. Weiss, in A. Papadopoulos (ed.), *Handbook of Teichmüller Theory*, Vol. VII, EMS Press, Zürich, 2020, 463–529.

[3] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.

[4] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, Cambridge University Press, Cambridge, 2005.

[5] M. Shishikura, *Conformality of quasiconformal mappings at a point, revisited*, Ann. Acad. Sci. Fenn. Math. **43** (2018), 981–990.